Where Did All of the Numbers Go?
Understanding and Succeeding in Math as MLLs: Mathematics Language Learners

Dr. Rachael Kenney
Can you solve this math problem?

Find the zamcof of a flapicle if the argorf is 12
• When we think of mathematics, we think of numbers, computations, rules and procedures.

• But language plays a key role in the understanding of mathematics

• Many students have difficulty understanding the language of mathematics
DEFINITION: language /ˈlaNGgwij/  

NOUN: A non-instinctive system of communication using symbols possessing arbitrary (conventional, learned) meanings and shared by a community.

A language contains the following components:

- There must be a **vocabulary** of words or symbols.
- **Meaning** must be attached to the words or symbols.
- Are **grammar** rules for how vocabulary is used.
- A **syntax** organizes symbols into linear structures.
- A group uses and understands the symbols.
Like other languages, Mathematics has nouns, pronouns, adjectives, verbs, and sentences.

**EXPRESSIONS** are the nouns and pronouns

**VARIABLES** represent, or hold the position of, mathematical objects.

An **EQUATION** is a declarative sentence in which the verb is “=”

**NOTATION/SYMBOLS** make up the writing system

The **SYNTAX** represents the rules of the language
Why do we use Symbols in Math?

"Symbols allow us to manipulate by proxy things that are not easily handled, or are impossible to handle, by our physical senses” (Arcavi, 1994, p. 109).

We can work easily and efficiently with symbols without paying much attention to their referents (Arcavi, 1994; Pimm 1995).

Using symbols as substitutes for abstract ideas is one of the great strengths of symbol use (Arcavi, 1994).
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Earliest use</th>
<th>First author to use</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>Plus sign</td>
<td>1360</td>
<td>Nicole Oresme</td>
</tr>
<tr>
<td>-</td>
<td>Minus sign</td>
<td>1489</td>
<td>Johannes Widmann</td>
</tr>
<tr>
<td>=</td>
<td>Equals sign</td>
<td>1557</td>
<td>Robert Recorde</td>
</tr>
<tr>
<td>×</td>
<td>Multiplication sign</td>
<td>1618</td>
<td>William Oughtred</td>
</tr>
<tr>
<td>√</td>
<td>Radical symbol</td>
<td>1637</td>
<td>René Descartes</td>
</tr>
<tr>
<td>∞</td>
<td>Infinity sign</td>
<td>1655</td>
<td>John Wallis</td>
</tr>
<tr>
<td>∫</td>
<td>Integral sign</td>
<td>1675</td>
<td>Gottfried Leibniz</td>
</tr>
<tr>
<td>∑</td>
<td>Summation symbol</td>
<td>1755</td>
<td>Leonhard Euler</td>
</tr>
<tr>
<td>!</td>
<td>Factorial</td>
<td>1808</td>
<td>Christian Kramp</td>
</tr>
<tr>
<td>ℵ</td>
<td>Aleph symbol</td>
<td>1893</td>
<td>Georg Cantor</td>
</tr>
<tr>
<td>∅</td>
<td>Empty set sign</td>
<td>1939</td>
<td>André Weil / Nicolas Bourbaki[2]</td>
</tr>
<tr>
<td>■</td>
<td>End of proof sign</td>
<td>1950</td>
<td>Paul Halmos</td>
</tr>
</tbody>
</table>
Some symbols came from simple abbreviations.
Some came from the first letters of words

Augustin-Louis Cauchy (1789-1857) used $\varepsilon$ and sometimes $\delta$ in 1821 in *Cours d'analyse*

$\delta$ meant 'différence' (French for *difference*), and

$\varepsilon$ meant 'erreur' (French for *error*)

(Finney & Thomas, 2002)
Some Evolved Over Time

From the Preface of Hardy’s 1908 book *A Course of Pure Mathematics* (p. ix):

I have followed Mr Leathem and Mr Bromwich in always writing
\[
\lim_{n \to \infty}, \lim_{x \to \infty}, \lim_{x \to a}
\]
and not \( \lim_{n=\infty}, \lim_{x=\infty}, \lim_{x=a} \). This last change seems to me one of considerable importance, especially when ‘\( \infty \)’ is the ‘limiting value’. I believe that to write ‘\( n=\infty, x=\infty \)’ (as if anything ever were ‘equal to infinity’), however convenient it may be at a later stage, is in the early stages of mathematical training to go out of one’s way to encourage incoherence and confusion of thought concerning the fundamental ideas of analysis.
Much of it came from getting tired of writing everything out in words!

Take some number that you are thinking of, multiply it by two, subtract one from the result, multiply the result of that by itself, divide the result of that by three, and then add one to get the final output.
An extract from Robert Recorde’s book “The Whetstone of Witte”

And to avoid the tedious repetition of these words: is equalle to: I will set as I doe often in work use, a pair of paralleles, or Gemowe lines of one length, thus: =====, because no 2 things, can be more equalle.”

Howbeit, for easie alteration of equations, I will propose a few examples, because the extraction of their roots, make the more aptly be wrought. And to a-noide the tedious repetition of these wordes: is equalle to: I will sette as I doe often in woork use, a paret of paralleles, or Gemowe lines of one lengthe, thus: =*=*=*, because noe 2 thinges, can be more equalle. And now marke these numbers.
Greetings everyone! Here is a radical idea (😃): For expressing equality, instead of writing “is equal to”, just use this symbol: 二
Why do we use the letter x for both multiplication and the most common variable?  

**We Don’t!**

• The multiplication symbol (×) is actually a representation of *the cross of Saint Andrew* (first used in the 16th century by English mathematician William Oughtred)

• Unable to sound out the Arabic term for “some thing”, Spanish scholars borrowed a similar sound from the Greeks – the letter “chi”. Over time, the Greek symbol morphed into its Latin doppelganger, x.
Variables for Unknowns

• Aristotle (384-322 BC) frequently used single capital letters or two letters for the designation of magnitude or number.

• Diophantus (250-275) used a Greek letter with an accent to represent an unknown. Xylander (1575) later translated Diophantus from Greek into Latin and used $N$ ($numerus$) for unknowns in equations.

• Brahmagupta (598-658) used the initials of the names of colors to represent unknowns in his equations in one of the earliest intimations of what we now know as algebra.
In 1637, René Descartes invented the convention of representing unknowns in equations by $x$, $y$, and $z$, and knowns by $a$, $b$, and $c$. 
How Letters are Used in Math Today

- **Constants**: $\pi$, $e$, $i$

- **Unspecified Constants**:
  - $a$, $b$, $c$, for real valued constants;
  - $n$ and $m$ for integer valued constants,
  - $p$ & $q$ for primes in number theory,

- **Unknowns**: $5x - 2 = 10$

- **Varying Quantities**: $y = 5x - 2$

- **Parameters**:
  - $y = mx + b$
  - $ax^2 + bx + c$

- **Real Variables**: $x$, $y$, $z$, $w$, $u$, $v$

- **Real Variables with Meanings**:
  - $t$ (time);
  - $\theta$ (angle);
  - $\lambda$ (eigenvalue)...

- **Integer Variables**: $i$, $j$, $k$

- **Complex Variables**: $z$; $a + bi$
Mathematics is a complex and compact symbol system; unless meanings are attached to those symbols, mathematics becomes literally meaningless to learners.

Mathematical language needs to be understood in relation to the speakers involved, the purpose of the communication, the audience, and the context.

Language is one of several resources students need and use to participate in mathematics thinking and learning.
\[
\begin{align*}
\ln(e) & \quad i \\
\frac{\partial y}{\partial x} & \quad V - E + F = ? \\
A & = \pi r^2 \\
\Delta x & \\
x = \sqrt[3]{bc - \frac{d}{2a} - \frac{b^3}{27a^3}} + \sqrt[3]{bc - \frac{d}{2a} - \frac{b^3}{27a^3}}^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3 \\
& + \sqrt[3]{bc - \frac{d}{2a} - \frac{b^3}{27a^3}} - \sqrt[3]{bc - \frac{d}{2a} - \frac{b^3}{27a^3}}^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3 - \frac{b}{3a} \\
\sum_{i=1}^{4} i
\end{align*}
\]
American Airlines says woman expressed suspicion about University of Pennsylvania economics professor, who turned out to be solving a differential equation.

Unfortunately, not everyone speaks this language easily
This can lead to fear of math and an aversion to interacting with mathematics symbols.
**Dual Nature of Symbolic Notation** (Tall et al, 2000)
The use of mathematical symbols enables mathematical concepts to be both a process and an object at the same time.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Process</th>
<th>Concept</th>
</tr>
</thead>
<tbody>
<tr>
<td>4+5</td>
<td>Addition</td>
<td>Sum</td>
</tr>
<tr>
<td>v= s/t</td>
<td>ratio</td>
<td>rate</td>
</tr>
<tr>
<td>f(x) =3+2x</td>
<td>Evaluation</td>
<td>Expression</td>
</tr>
<tr>
<td></td>
<td>Assignment</td>
<td>Function</td>
</tr>
<tr>
<td>dy/dx</td>
<td>Differentiation</td>
<td>Derivative</td>
</tr>
</tbody>
</table>
Proceptual thinking: “The ability to compress stages in symbol manipulation to the point where symbols are viewed as objects that can be decomposed and recomposed in flexible ways”

(Gray & Tall, 1994, p. 132).
Ex. $\log_5(x) + \log_5(x+4)$

<table>
<thead>
<tr>
<th>Process</th>
<th>Composition into a new object using the laws of logarithms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concept</td>
<td>Inverse relationship for exponentials</td>
</tr>
<tr>
<td></td>
<td>Log notation leaves students “bereft of a succinct way to verbalize the operation performed on the input” (Hurwitz, 1999, p. 344)</td>
</tr>
</tbody>
</table>
Two students were interviewed while solving this task:

\[
\text{Solve } \log_7(2x + 1) = 2
\]

Both students wrote \(7^2 = 2x + 1\), and solved for \(x\) correctly to solve the problem.

- Does this indicate a proceptual understanding of a logarithmic problem presented in this form?
The same two students also solved this problem

Solve $\log_5(x) + \log_5(x + 4) = 1$

**Student A:**

$$x + x + 4 = 1$$

I: What happened to log?
L: I guess they cancel out.
I: What does that mean to you?
L: I guess it's like when you have a negative 1, or a 1 and a negative 1. That would give you zero, so that's how they cancel out.

**Student B:**

$$x(x + 4) = 1$$

I: So what happened to log?
A: They cancel out.
I: What does canceling out mean to you?
A: They have a, I don't know if complementary is the right word, but another one was there...so that positive or negative or something like that. Two of something.
Students who struggle with mathematics are not learning correct mathematics more slowly – they are actually developing different techniques for problem solving due to their interpretation of the symbols involved?
Ex. 2 – Rational Equation

Solve for \( x \):

\[
\frac{x - 16}{x^2 - 3x - 12} = 0
\]
What did the Students “see” in the symbols?

<table>
<thead>
<tr>
<th>$\frac{x-16}{(x^2-3x-12)} \cdot \frac{x^2-3x-12}{(x^2-3x-12)}$</th>
<th>$\frac{x-16 = 0}{(x\quad )(x\quad )}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 - 3x - 12 = 0$</td>
<td>$(x + 4)(x - 4)$</td>
</tr>
<tr>
<td></td>
<td>$x(x - 3) - 12$</td>
</tr>
</tbody>
</table>

They saw a need to **factor** and/or **get rid of the fraction**
The Reversal Error

“Write an equation using the variables $S$ and $P$ to represent the following statement: ‘There are six times as many students as professors at this university.’ Use $S$ for the number of students and $P$ for the number of professors.”

Reversing the relationship between two variables in a mathematical word problem is a long-standing mathematical obstacle frequently encountered by students at all education levels (e.g., Clement, 1982; K. M. Fisher, 1988c; Philipp, 1992; Weinberg, 2009; Kim et al., 2012).
Potential Causes

1. Direct translation: matching the sequence of algebraic symbols and the sequence of objects in a word problem. (Clement, 1982; Fisher, Borchert, Bassok 2010)

   **There are 6 times as many students as professors**
   means $6 \times S = P$

2. Use of a static comparison approach (Cohen & Kanim, 2005; Stacey & McGregor, 1993)

   - Variables $S$ and $P$ are abbreviations for “Student” and Professor” rather than “Number of students/professors”

   **Every 6 students for 1 professor means** $6S = P$
The language of mathematics differs from ordinary speech in three important ways:

- It is non-temporal – there is no past, present or future
- It is devoid of emotional content
- It is precise and adverse to ambiguity...
...Except when it is ambiguous!

Problematic language includes:

- Terms like Bigger Half (or constant variable? Random pattern? Exact approximation?...)

- Reduce the fraction \( \frac{6}{12} = \frac{3}{6} = \frac{1}{2} \)

- Inverse – inverse function? Inverse operation, multiplicative inverse, additive inverse, inverse matrix, inverse variation...???
Many ways to indicate multiplication

12 \cdot 3

12 \times 3

(12)(3)

12(3)

12x

3!

xyz

f(x)?
A question posed is, why is the language of mathematics confusing to students? The short answer is because we, consciously or unconsciously, make it confusing.

(Bulaon, 2018)

\[ 5(a + b) = 5a + 5b, \text{ but } \sin(a + b) \neq \sin(a) + \sin(b) \]

\[(2, 7) \text{ is a point } \]
\[(2, 7) \text{ is an interval } \]
\[1 + (2-7)^2 \text{ indicates order of operation} \]

\[ \frac{dx}{dy} \text{ does not represent division!} \]

\[ f^{-1}(x) \neq (f(x))^{-1} \]
Some see mathematics as the language of the universe simply because it has to be - There is no simpler, more fundamental way of expressing the universe.

Every culture uses principals of math to help with everyday life from business to politics and these principals remain the same.
Mathematics – The Not-So-Universal Language?

23,467,891,705
“23 billion, 467 million, 891 thousand, 705” (US)

“23 thousand million, 467 million, 891 thousand, 705” (UK and Latin America)

23 mil millones 467 millones, 891 mil, 705 (Spain)
• In English, you read from left to right.
• In Arabic, you read right to left.
• In Japanese, you read top to bottom and right to left!
• In mathematics, order varies based on the operations involved
Polysemous Words =

- Table
- Set
- Normal
- Power
- Imaginary
- Odd
- Order
- Operation

- Plot
- Prime
- Rational
- Even
- Mean
- Volume
- Range
- Negative

“Mathematics is the art of giving the same name to different things.”
– Henri Poincaré
Homophones can cause students great difficulties with understanding the rapid speech of lectures

- before/four/for
- eight/ate
- exchange/change
- in/an
- know/no
- many/money
- sum/some
- tenths/tents
- two/to/too
- whole-hole
Some words are related, but their distinct meanings are often confused (Thompson & Rubenstein, 2000)
Explain the difference between the directions “Solve, Evaluate, and Simplify”

I think we are so conditioned to know what to look for we don't look at the directions unless we are confused... If [it says] \(3(x+2)-x=0\), we'd solve.

I never realized how much informal language I use, and how I often find myself neglecting the more formal language of mathematics.
Describe everything you understand about mathematical functions

When reading this prompt, I knew right away that I do not know as much mathematical vocabulary as I need to know. If you asked me to graph a function, or integrate a function, I would have no trouble. But, when asked to define a function, my mind was blank.

After thinking about this prompt for a few days I began to realize that I know a lot about functions, but I'm not able to explain much about functions.
At the sentence level, there are language patterns and grammatical structures that are unique to the language of mathematics.

The vocabulary in mathematics is vast; we need deliberate, systematic, and contextualized instruction for students to engage in mathematical thinking and learning.

Learners need opportunities to show how they are processing the language they are using in the mathematics classroom.

We often make mistakes not because we do not know what operation to do, but because we misunderstand the meaning of one single word. This is an issue we need to be aware of for all learners.